ICARE - localising Conditional AutoRegressive Expectiles

Wolfgang Karl Härdle
Xiu Xu
Andrija Mihoci

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics
and Economics
Humboldt–Universität zu Berlin
Brandenburg University of Technology
lvb.wiwi.hu-berlin.de
case.hu-berlin.de
irtg1792.hu-berlin.de
b-tu.de

Motivation

- Risk Exposure
  - Measure tail events
  - Conditional autoregressive expectile (CARE) model ▸ Expectiles
- Time-varying parameters
  - Time-varying parameters in CARE ▸ Parameter Dynamics
  - Interval length reflects the structural changes in economy

Objectives

- Localising CARE Models
  - Local parametric approach (LPA)
  - Balance between modelling bias and parameter variability
- Tail Risk Dynamics
  - Estimation windows with varying lengths
  - Time-varying expectile parameters

Econometrics and Risk Management

Econometrics

- Modelling bias vs. parameter variability
- Interval length and economic variables

Risk Management

- Parameter dynamics and structural changes
- Measuring tail risk

Risk Exposure

An investor observes daily DAX returns from 20050103 to 20141231 and estimates the underlying risk exposure via expectiles (e.g., 1% and 5%) over a one-year time horizon.

Modelling strategies

(a) Data windows fixed on an ad hoc basis

(b) Adaptively selected data intervals: time-varying parameters

Portfolio Protection

An investor decides about the daily allocation into a stock portfolio (DAX). Goal: a proportion of the initial portfolio value (100) is preserved at the end of a horizon, i.e., the target floor equals 90.

Decision at day t: multiple of the difference between the portfolio value and the discounted floor up to t is invested into the stock portfolio (DAX), the rest into a riskless asset.

Multiplier m selection: constant or time-varying (ICARE) ▸ Constant m

Research Questions

How to account for time-varying parameters in tail event risk measures estimation?

What are the typical data interval lengths assessing risk more accurately, i.e., striking a balance between bias and variability?

How well does the ICARE technique perform in practice?

Outline

1. Motivation ✓
2. Conditional Autoregressive Expectile (CARE)
3. Local Parametric Approach (LPA)
4. Empirical Results
5. Applications
6. Conclusions

Conditional Autoregressive Expectile

- Taylor (2008), Kuan et al. (2009), Engle and Manganelli (2004) ▸ CAViaR
- Random variable Y (e.g. returns), identically distributed, y_t, $t = 1, ..., n$
- CARE specification conditional on information set $\mathcal{F}_{t-1}$

$$y_t = e_{t,\tau} + \varepsilon_{t,\tau}$$

▸ $\varepsilon_\tau \sim \text{AND}\left(0, \sigma^2_{\varepsilon,\tau}, \tau\right)$

$$e_{t,\tau} = \alpha_{0,\tau} + \alpha_{1,\tau} y_{t-1} + \alpha_{2,\tau}\left(y^+_{t-1}\right)^2 + \alpha_{3,\tau}\left(y^-_{t-1}\right)^2$$

- Expectile $e_{t,\tau}$ at $\tau \in (0,1)$, $\theta_\tau = \left\{\alpha_{0,\tau}, \alpha_{1,\tau}, \alpha_{2,\tau}, \alpha_{3,\tau}, \sigma^2_{\varepsilon,\tau}\right\}^\top$
- Returns: $y^+_{t-1} = \max\{y_{t-1}, 0\}$, $y^-_{t-1} = \min\{y_{t-1}, 0\}$

Parameter Estimation

- Data calibration with time-varying intervals
- Observed returns $\mathcal{Y} = \{y_1, \dots, y_n\}$
- Quasi maximum likelihood estimate (QMLE)

$$\widetilde{\theta}_{I,\tau} = \arg\max_{\theta_\tau \in \Theta} \ell_I\left(\mathcal{Y}; \theta_\tau\right)$$

▸ $\ell_I(\cdot)$

- $I = [t_0 - v, t_0]$ - interval of $(v+1)$ observations at t_0
- $\ell_I(\cdot)$ - quasi log likelihood

Estimation Quality

- Mercurio and Spokoiny (2004), Spokoiny (2009)
- Quality of estimating *true* parameter vector θ^*_τ by QMLE $\widetilde{\theta}_{I,\tau}$ in terms of Kullback-Leibler divergence; $\mathcal{R}_r\left(\theta^*_\tau\right)$ - risk bound

$$\mathsf{E}_{\theta^*_\tau}\left|\ell_I(\mathcal{Y};\widetilde{\theta}_{I,\tau})-\ell_I(\mathcal{Y};\theta^*_\tau)\right|^r \leq \mathcal{R}_r\left(\theta^*_\tau\right)$$

▸ $\mathcal{R}_r\left(\theta^*_\tau\right)$ ▸ Gaussian Regression

- 'Modest' risk, $r = 0.5$ (shorter intervals of homogeneity)
- 'Conservative' risk, $r = 1$ (longer intervals of homogeneity)

Solomon Kullback and Richard A. Leibler on BBI:

Local Parametric Approach (LPA)

- LPA, Spokoiny (1998, 2009)
  - Time series parameters can be locally approximated
  - Finding the *interval of homogeneity* Details
  - Balance between modelling bias and parameter variability

- Time series literature
  - GARCH(1, 1) models - Čížek et al. (2009)
  - Realized volatility - Chen et al. (2010)
  - Multiplicative Error Models - Härdle et al. (2015)

Interval Selection

- $(K+1)$ nested intervals with length $n_k = |I_k|$

$$\begin{array}{ccccccc} I_0 & \subset & I_1 & \subset \cdots \subset & I_k & \subset \cdots \subset & I_K \\ \widetilde{\boldsymbol{\theta}}_0 & & \widetilde{\boldsymbol{\theta}}_1 & & \widetilde{\boldsymbol{\theta}}_k & & \widetilde{\boldsymbol{\theta}}_K \end{array}$$

Example: Daily index returns

Fix t_0, $I_k = [t_0 - n_k, t_0]$, $n_k = \left[n_0 c^k\right]$, $c > 1$

$\{n_k\}_{k=0}^{11} = \{20 \text{ days}, 25 \text{ days}, \ldots, 250 \text{ days}\}$, $c = 1.25$

Local Change Point Detection

- Fix t_0, sequential test $(k = 1, \ldots, K)$

H_0 : parameter homogeneity within I_k
H_1 : $\exists$ change point within $J_k = I_k \setminus I_{k-1}$

$$T_{k,\tau} = \sup_{s \in J_k} \left\{ \ell_{A_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{A_{k,s},\tau} \right) + \ell_{B_{k,s}} \left(\mathcal{Y}, \widetilde{\theta}_{B_{k,s},\tau} \right) - \ell_{I_{k+1}} \left(\mathcal{Y}, \widetilde{\theta}_{I_{k+1},\tau} \right) \right\}$$

with $A_{k,s} = [t_0 - n_{k+1}, s]$ and $B_{k,s} = (s, t_0]$

Critical Values, $\mathfrak{z}_{k,\tau}$

- Simulate $\mathfrak{z}_k$ - homogeneity of the interval sequence $I_1, \ldots, I_k$
- 'Propagation' condition

$$\mathsf{E}_{\theta^*_\tau} \left| \ell_{I_k}\left(\mathcal{Y}; \widetilde{\theta}_{I_k,\tau}\right) - \ell_{I_k}\left(\mathcal{Y}; \widehat{\theta}_\tau\right) \right|^r \leq \rho_k \mathcal{R}_r\left(\theta^*_\tau\right)$$

$\rho_k = \dfrac{\rho k}{K}$ for a given significance level ρ ▸ $\widehat{\theta}_\tau$ - adaptive estimate

- Check $\mathfrak{z}_{k,\tau}$ for (six) different θ^*_τ ▸ Parameter Scenarios

Critical Values, $\mathfrak{z}_{k,\tau}$

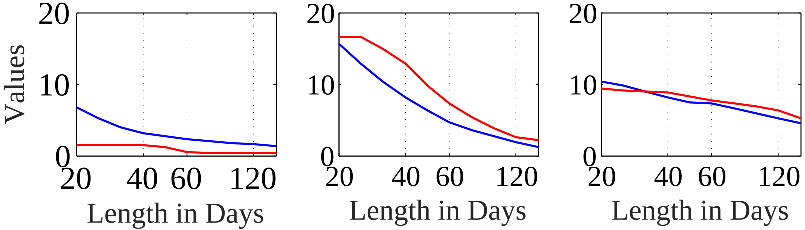


Figure 1: Simulated critical values across different parameter constellations ▸ Parameter Scenarios for the modest case $r = 0.5$, $\tau = 0.05$ and $\tau = 0.01$

Critical Values, $\mathfrak{z}_{k,\tau}$

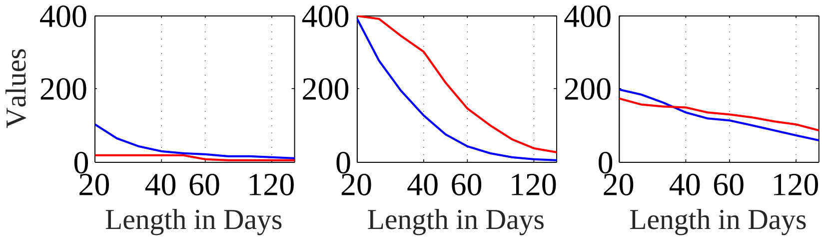


Figure 2: Simulated critical values across different parameter constellations ▸ Parameter Scenarios for the conservative case $r = 1$, $\tau = 0.05$ and $\tau = 0.01$

Adaptive Estimation

▸ LPA ▸ $\mathfrak{z}_{k,\tau}$ - Critical Values

- Compare $T_{k,\tau}$ at every step k with $\mathfrak{z}_{k,\tau}$
- Data window index of the *interval of homogeneity* - $\widehat{k}$

- Adaptive estimate

$$\widehat{\theta}_{\tau} = \widetilde{\theta}_{I_{\hat{k}},\tau}, \quad \widehat{k} = \max_{k \leq K} \{k : T_{\ell,\tau} \leq \mathfrak{z}_{\ell,\tau}, \ell \leq k\}$$

Data

- Series
  - DAX, FTSE 100 and S&P 500 returns 20050103-20141231 (2608 days)
  - Research Data Center (RDC) - Datastream
- Setup
  - Expectile levels: $\tau = 0.05$ and $\tau = 0.01$
  - Modest ($r = 0.5$) and conservative ($r = 1$) risk cases
  - $\{n_k\}_{k=0}^{11} = \{20 \text{ days}, 25 \text{ days}, \ldots, 250 \text{ days}\}$

Adaptive Estimation

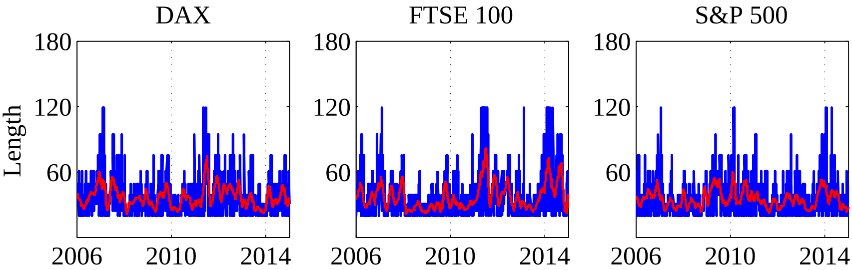


Figure 3: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the modest risk case $r = 0.5$, at expectile level $\tau = 0.05$. The red line presents the one-month smoothed values. ▸ Parameter Flag

Adaptive Estimation

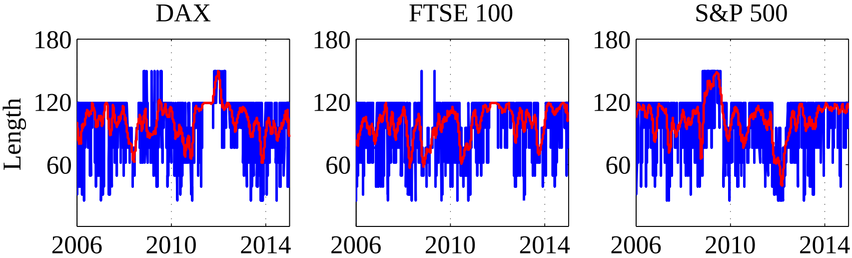


Figure 4: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the conservative risk case $r = 1$, at expectile level $\tau = 0.05$. The red line presents the one-month smoothed values. ▸ Parameter Flag

Adaptive Estimation

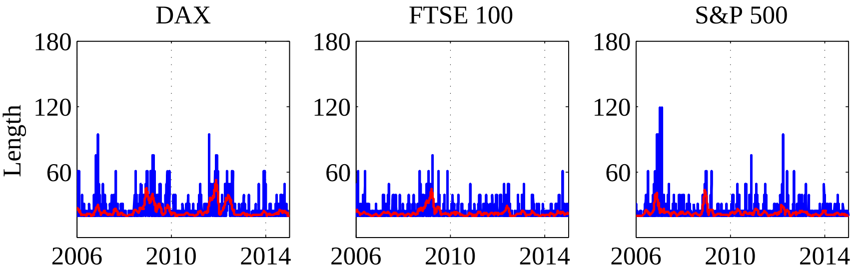


Figure 5: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the modest risk case $r = 0.5$, at expectile level $\tau = 0.01$. The red line presents the one-month smoothed values. ▸ Parameter Flag

Adaptive Estimation

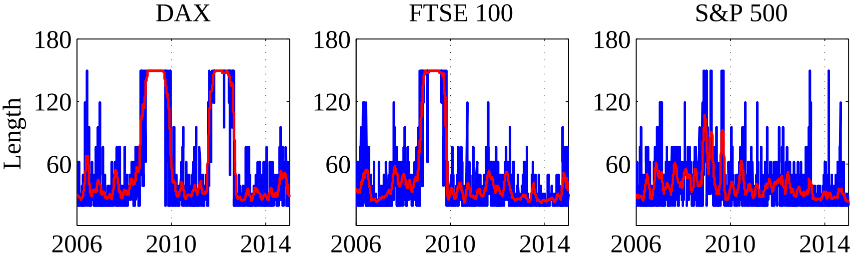


Figure 6: Estimated length $n_{\hat{k}}$ of *intervals of homogeneity* from 20060103-20141231 for the conservative risk case $r = 1$, at expectile level $\tau = 0.01$. The red line presents the one-month smoothed values. ▸ Parameter Flag

Risk Exposure

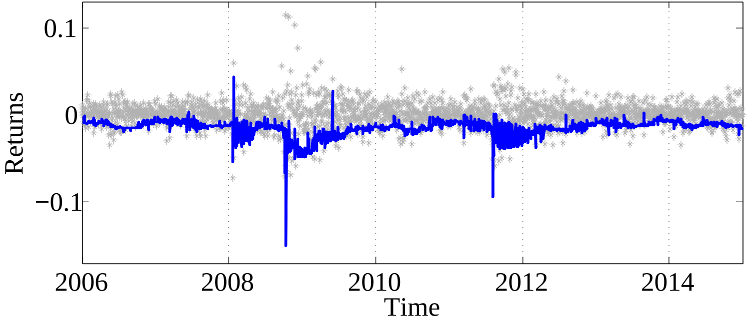
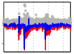


Figure 7: DAX index returns ($*$) and adaptively estimated expectile $e_{t,\tau}$ ($r = 1$ and $\tau = 0.05$) from 20060103-20141231



Risk Exposure

▸ Expected Shortfall $ES_{e_{t,\tau}}$

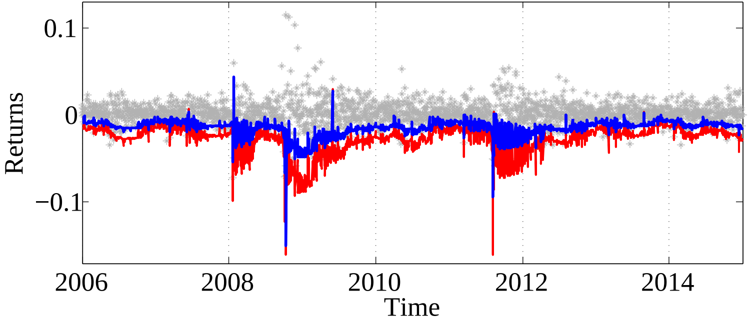


Figure 8: DAX index returns ($*$), adaptively estimated expectile $e_{t,\tau}$ and expected shortfall $ES_{e_{t,\tau}}$ ($r = 1$ and $\tau = 0.05$) from 20060103-20141231

Portfolio Protection

- Portfolio protection strategy ▸ Details
  - Aim: Guarantee a proportion level of wealth at the investment horizon.
  - The investor can reduce the downside risk as well as participating in gains of risky assets.

Example

Decision at day t: multiple of the difference between the portfolio value and the discounted floor up to t is invested into the stock portfolio (DAX), the rest into a riskless asset

Portfolio Protection

- Crucial ingredient: the multiplier m
  - m: the proportion value invested into risky assets
  - The larger m, the more risky exposure
- How to select the multiplier?
  - Standard constant value Constant m
  - Based on tail risk measure, VaR or ES Details
- Multiplier selection - Hamidi et al. (2014), ICARE

$$m_{t,\tau} = \left| ES_{e_{t,\tau}} \right|^{-1}$$

  - Practice: threshold range for $m_{t,\tau}$, $[1, 12]$

Multiplier Dynamics

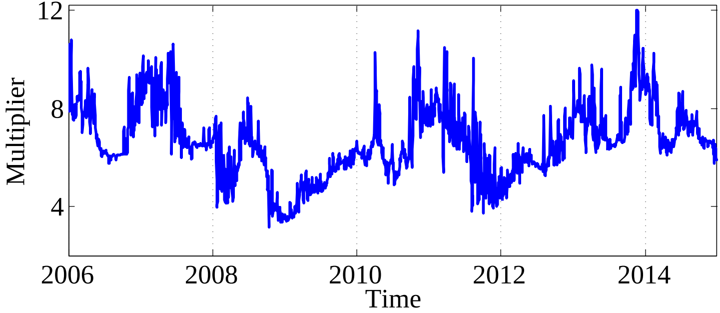


Figure 9: Time-varying multiplier $m_{t,\tau}$ for DAX index returns based on ICARE ($r = 1$ and $\tau = 0.05$) from 20060103-20141231 ▸ Multiplier Density

Performance

▸ One-year rolling details ▸ CAViaR-based rolling details ▸ Target floor

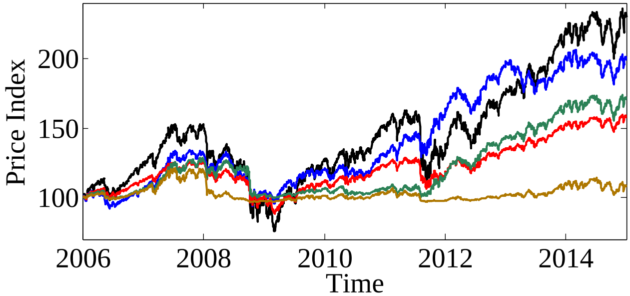


Figure 10: Portfolio value: (a) DAX index (black), (b) $m = 5$, (c) one-year rolling , (d) CAViaR one-year rolling ($\alpha = 0.065$), (e) $m_{t,\tau}$ - ICARE ($r = 1$ and $\tau = 0.05$) from 20060103-20141231.

Performance

	Return(%)	Volatility(%)	VaR 99%	Skewness	Kurtosis	Sharpe
Data	8.79	22.54	-4.24	0.24	10.33	0.02
lCARE	7.36	13.60	-2.31	0.52	9.16	0.03
Rolling one-year	5.70	10.18	-1.59	0.17	10.05	0.04
CAViaR rolling	0.01	7.35	-1.43	-0.90	13.04	0.00
Multiplier 1	3.51	2.25	-0.41	0.20	10.05	0.10
Multiplier 2	3.97	4.50	-0.84	0.19	10.00	0.06
Multiplier 3	4.41	6.74	-1.27	0.17	9.90	0.04
Multiplier 4	4.78	9.00	-1.71	0.15	9.88	0.03
Multiplier 5	4.86	11.17	-2.10	0.11	9.91	0.03
Multiplier 6	3.36	5.36	-0.99	-0.33	6.48	0.04
Multiplier 7	2.65	6.04	-1.08	-0.51	6.49	0.03
Multiplier 8	2.13	6.55	-1.17	-0.59	7.90	0.02
Multiplier 9	1.70	6.96	-1.25	-0.74	10.38	0.02
Multiplier 10	1.46	7.33	-1.38	-0.93	12.90	0.01
Multiplier 12	0.82	7.56	-1.47	-1.25	16.65	0.01

Figure 11: Portfolio return moments comparison. Returns and volatility are annualized. The investment strategy is on a one-year investment basis.

Performance - Summary

- ICARE vs empirical data
  - Slightly lower return (7.36% vs 8.79%)
    largely lower volatility (13.60% vs 22.54%)
  - Guarantee the target floor value
- ICARE vs other strategies
  - higher return than the candidates with CAViaR-based or expectile one-year rolling
  - Outperform typical constant multiplier benchmarks

Conclusions

- Localising CARE Model
  - Balance between modelling bias and parameter variability
  - Parameter dynamics
- Tail Risk Dynamics
  - Expectile levels $\tau = 0.05$ and $\tau = 0.01$
  - Expectile and Expected Shortfall
- Asset Allocation
  - Portfolio insurance on DAX at level $\tau = 0.05$
  - Outperform one-year rolling window and other benchmarks

ICARE - localising Conditional AutoRegressive Expectiles

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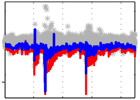

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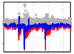

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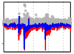


Why Expectiles? Quantile VaR

▸ Motivation

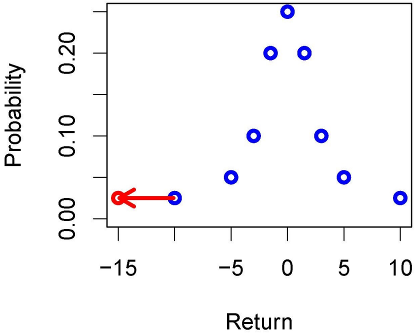


Figure 12: Distribution of returns, the 5% quantile remains unchanged under the changing tail structure

Expectile v.s. Quantile

Motivation

- Tail inference
  - Quantile: zero-moment of tail structure - probability
    Central quantile: median
  - Expectile: first moment of tail structure
    Central expectile: mean
- Expectiles are sensitive to extreme magnitude, outliers
- Expectiles link to expected shortfall (ES) nicely

M-Quantiles

▸ Motivation

- Loss function, Breckling and Chambers (1988)

$$z_\alpha = \arg\min_\theta \mathsf{E}\, \rho_{\alpha,\gamma}(Y - \theta)$$

where $\rho_{\alpha,\gamma}(u) = |\alpha - \{u < 0\}|\, |u|^\gamma, \quad \gamma \geq 1$

  - Quantile - ALD location estimate

$$q_\alpha = \arg\min_\theta \mathsf{E}\, \rho_{\alpha,1}(Y - \theta)$$

  - Expectile - AND location estimate

$$e_\alpha = \arg\min_\theta \mathsf{E}\, \rho_{\alpha,2}(Y - \theta)$$

Loss Function

▸ Motivation

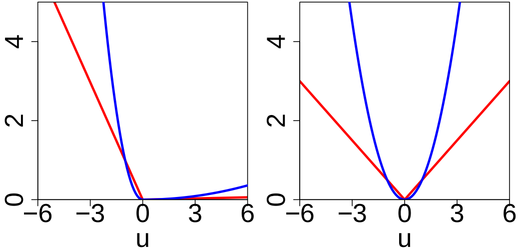


Figure 13: Expectile and quantile loss functions at $\alpha = 0.01$ (left) and $\alpha = 0.50$ (right)

LQRcheck

Expectiles and Quantiles

Motivation

- M-Quantile

$$\frac{\alpha}{1-\alpha} = \frac{\int_{-\infty}^{e_\alpha} |y - e_\alpha|^{\gamma-1} \, dF(y)}{\int_{e_\alpha}^{\infty} |y - e_\alpha|^{\gamma-1} \, dF(y)}$$

  - Expectile - Global influence, obtained from

$$\gamma = 2, \qquad \frac{\alpha}{1-\alpha} = \frac{\int_{-\infty}^{e_\alpha} |y - e_\alpha| \, dF(y)}{\int_{e_\alpha}^{\infty} |y - e_\alpha| \, dF(y)}$$

  - Quantile - Local influence, obtained from

$$\gamma = 1, \qquad \frac{\alpha}{1-\alpha} = \frac{\mathrm{P}\left(Y \le q_\alpha\right)}{\mathrm{P}\left(Y > q_\alpha\right)}$$

CAViaR - Conditional Autoregressive Value at Risk by Regression Quantiles

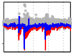
CARE CAViaR performance

- Engle and Manganelli (2004)
- Asymmetric slope specification, conditional on information set $\mathcal{F}_{t-1}$ at time t

$$y_t = q_{t,\alpha} + \varepsilon_{t,\alpha} \qquad Quant_\alpha(\varepsilon_{t,\alpha}|\mathcal{F}_{t-1}) = 0$$
$$q_{t,\alpha} = \beta_0 + \beta_1 q_{t-1,\alpha} + \beta_2 y^+_{t-1} + \beta_3 y^-_{t-1}$$

- Quantile (VaR) $q_{t,\alpha}$ at $\alpha \in (0,1)$, $Quant_\alpha(\varepsilon_{t,\alpha}|\mathcal{F}_{t-1})$ is the α-quantile of $\varepsilon_{t,\alpha}$ conditional on information set $\mathcal{F}_{t-1}$
- With AND, set $\alpha = 0.065$ such that $e_{\tau_\alpha} = q_\alpha$ when $\tau_\alpha = 0.05$

Details

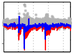


Asymmetric Normal Distribution (AND) ▸ CARE

- AND $\left(\mu, \sigma^2, \tau\right)$ pdf:

$$f\,(w) = \frac{2}{\sigma}\left(\sqrt{\frac{\pi}{|\tau-1|}} + \sqrt{\frac{\pi}{\tau}}\right)^{-1} \exp\left\{-\rho_\tau\left(\frac{w-\mu}{\sigma}\right)\right\}$$

  - Check function: $\rho_\tau\,(u) = |\tau - \mathbf{I}\,\{u \le 0\}|\,u^2$
  - AND $\left(\mu, \sigma^2, 1/2\right) = \mathrm{N}(\mu, \sigma^2)$, Gerlach et al. (2012)



PDF

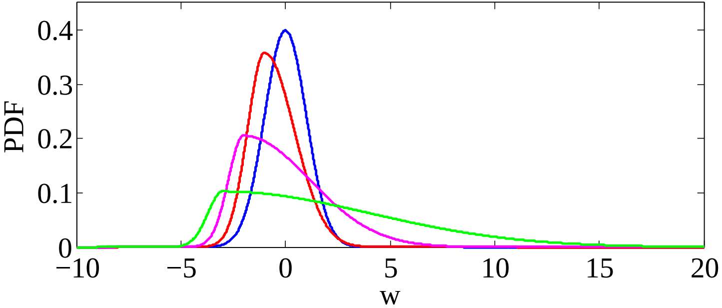
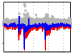


Figure 14: Density function for selected ANDs: (a) $\mu = 0, \tau = 0.5$ (b) $\mu = -1, \tau = 0.25$ (c) $\mu = -2, \tau = 0.05$ (d) $\mu = -3, \tau = 0.01$, with $\sigma^2_{\varepsilon_\tau} = 1$



Quasi Log Likelihood Function

▸ Parameter Estimation

- If $\varepsilon_\tau \sim \text{AND}\left(\mu, \sigma_\varepsilon^2, \tau\right)$ with pdf $f_\varepsilon(\cdot)$
  then $Y \sim \text{AND}\left(e_\tau + \mu, \sigma_\varepsilon^2, \tau\right)$
- Quasi log likelihood function for observed data
  $\mathcal{Y} = \{y_1, \ldots, y_n\}$ over a fixed interval I

$$\ell_I(\mathcal{Y}; \theta_\tau) = \sum_{t \in I} \log f_\varepsilon(y_t - e_{t,\tau})$$

Gaussian Regression

Estimation Quality

$$Y_i = f(X_i) + \varepsilon_i,\, i = 1, \dots, n, \text{ weights } W = \{w_i\}_{i=1}^n$$

$$L(W,\theta) = \sum_{i=1}^n \ell\{Y_i, f_\theta(X_i)\} w_i, \text{ log-density } \ell(\cdot),\ \widetilde{\theta} = \arg\max_{\theta \in \Theta} L(W,\theta)$$

1. Local constant, $f(X_i) \approx \theta^*$, $\varepsilon_i \sim \mathrm{N}(0, \sigma^2)$

$$\mathrm{E}_{\theta^*} \left| L(W,\widetilde{\theta}) - L(W,\theta^*) \right|^r \leq \mathrm{E}\,|\xi|^{2r}, \quad \xi \sim \mathrm{N}(0,1)$$

2. Local linear, $f(X_i) \approx \theta^{*\top}\Psi_i$, $\varepsilon_i \sim \mathrm{N}(0, \sigma^2)$, basis functions $\Psi = \{\psi_1(X_1), \dots, \psi_p(X_p)\}$, multivariate ξ

$$\mathrm{E}_{\theta^*} \left| L(W,\widetilde{\theta}) - L(W,\theta^*) \right|^r \leq \mathrm{E}\,|\xi|^{2r}, \quad \xi \sim \mathrm{N}(0,\mathcal{I}_p)$$

Risk Bound

▸ Estimation Quality

	$\tau = 0.05$			$\tau = 0.01$		
	Low	Mid	High	Low	Mid	High
$r = 0.5$	0.24	0.33	0.25	0.38	0.38	0.15
$r = 1.0$	2.40	4.62	2.75	5.90	5.81	1.15

Table 1: Simulated $\mathcal{R}_r(\theta^*_\tau)$, with expectile levels $\tau = 0.05$ and $\tau = 0.01$, for six selected parameter constellation groups ▸ Parameter Scenarios

Parameter Scenarios

▸ Risk Bound ▸ Critical Values

	$\tau = 0.05$			$\tau = 0.01$		
	Low	Mid	High	Low	Mid	High
$\widetilde{\alpha}_{0,\tau}$	-0.0003	0.0003	0.0007	-0.0003	0.0003	0.0007
$\widetilde{\alpha}_{1,\tau}$	-0.1058	-0.0306	0.0524	-0.1035	-0.0312	0.0547
$\widetilde{\alpha}_{2,\tau}$	-0.5800	-0.5288	0.2438	-0.5808	-0.5266	0.2089
$\widetilde{\alpha}_{3,\tau}$	0.5050	0.5852	2.1213	0.5134	0.5871	2.2066
$\widetilde{\sigma}^2_{\varepsilon,\tau}$	0.0001	0.0001	0.0002	0.0001	0.0001	0.0002

Table 2: Quartiles of estimated CARE parameters based on one-year estimation window, i.e., 250 observations, for the three stock market returns - DAX, FTSE 100, S&P 500 - from 20050103-20141231 (2608 trading days)

Selected Parameter Scenarios

Adaptive Estimation

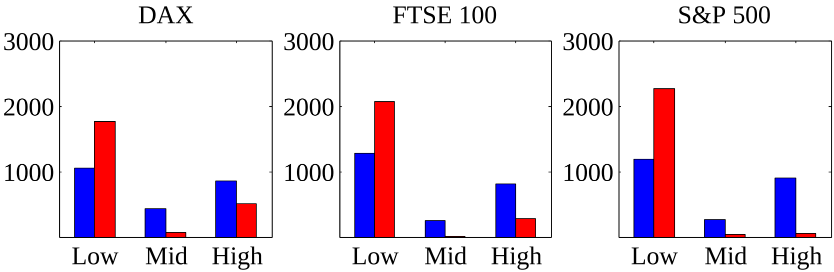


Figure 15: Histogram of the selected parameter scenarios (Low, Mid and High) for adaptive estimation with $\tau = 0.05$ and $\tau = 0.01$.

Expected Shortfall

▸ Risk Exposure ▸ CAViaR

- Expectile level τ_α such that $e_{\tau_\alpha} = q_\alpha$ (α-quantile), Yao and Tong (1996), Acerbi and Tasche (2002)

$$\tau_\alpha = \frac{\alpha \cdot q_\alpha - \int_{-\infty}^{q_\alpha} y dF(y)}{\mathsf{E}[Y] - 2\int_{-\infty}^{q_\alpha} y dF(y) - (1-2\alpha)\, q_\alpha}$$

where $Y \sim AND$.

- Expected Shortfall (ES), Kuan et al. (2009)

$$ES_{e_{\tau_\alpha}} = \left| 1 + \tau_\alpha \left(1 - 2\tau_\alpha\right)^{-1} \alpha^{-1} \right| e_{\tau_\alpha}$$

Portfolio Protection Strategy

▸ Strategy ▸ Multiplier

- Under certain confidence level, we aim to maintain: Estep and Kritzman (1988)

$$V_t \geq \ k \times \max\left\{F * e^{-rf*(T-t)}, \sup_{p\leq t} V_p\right\} = F_t^s$$

  - V_t: portfolio value at time t, $t \in (0, T]$
    F_t^s: protection value (target floor)
  - k exogenous parameter $(0, 1)$, set $k = 0.9$
    rf risky free rate, initial value $F = 100$
  - Cushion value $C_t = V_t - F_t^s \geq 0$

- Allocate $G_t = m \cdot C_t$ proportion into stock portfolio (DAX), and the remaining $V_t - G_t$ into riskless asset, multiplier $m \geq 0$.

Example: CPPI - Constant proportion portfolio insurance

Consider an insurance strategy under CPPI with constant floor $F = 100$, constant $m = 5$, and riskless asset rate $rf = 0$ (cash). The initial risky asset value is 100, and at each step goes up(down) 15.

initial risky asset value F	100
proportion k	0.9
riskless rate rf	0
constant multiplier m	5
steps	4

Risky asset value				
				160
			145	(0.10)
		130	(0.12)	130
	115	(0.13)	115	(0.13)
100	(0.15)	100	(0.15)	100
	85	(0.18)	85	(0.18)
	(-0.15)	70	(0.21)	70
		(-0.18)	55	(0.27)
			(-0.21)	40
				(-0.27)

Table 3: Risky portfolio value and the value in low bracket denotes the asset return.

Portfolio value and the cushion				
				159.18
			135.59	(69.18)
		118.91	(45.60)	103.61
	107.50	(28.91)	98.24	(13.61)
100	(17.5)	94.71	(8.24)	91.15
(10)	92.50	(4.71)	90.61	(1.15)
	(2.5)	90.29	(0.61)	89.979
		(0.29)	89.979	(0)
			(-0.02)	89.979
				(0)

Table 4: Portfolio value and the value in low bracket denotes the cushion.

Multiplier

▸ Multiplier

- Portfolio value V_t

$$V_{t+1} = V_t + G_t r_{t+1} + (V_t - G_t)\, rf_{t+1}$$

with r_t stock index return and rf_t riskless rate

- Cushion value $C_t = V_t - F_t^s \geq 0$

$$C_{t+1} = C_t\{1 + m \cdot r_{t+1} + (1 - m)\, rf_{t+1}\}$$

- $\forall t \leq T$, since the value $C_t \geq 0$

$$m \cdot r_{t+1} + (1 - m)\, rf_{t+1} \geq -1$$

Multiplier

▸ Multiplier ▸ Gap risk

- If rf_t is relatively small, and when $r_{t+1} < 0$, yield the upper bound on the multiplier:

  Proposition The guarantee is satisfied at any time of the management period with a probability equal to 1 ✧

  $$\forall t \leq T - 1, m \leq \left(-r_{t+1}^{-}\right)^{-1}$$

  where $r_{t+1}^{-} = \min\{r_{t+1}, 0\}$.

  - Multiplier m_t, the leverage value on risky assets, is negatively related to the maximum extreme loss of risky assets.
  - For example, if $r_{t+1} = -10\%$, $m \leq 10$; If $r_{t+1} = -20\%$, $m \leq 5$.

Gap Risk

▸ Multiplier

- In practice, due to the discrete-time rebalancing, the nonnegative cushion value can not be guaranteed perfectly.
▸ Details
- Gap risk: the risk of violating the floor protection, i.e., the tiny level of probability that the cushion values are non-positive.
- How to define the gap risk:
  - control of the probability of a potential loss - VaR based multiplier
  - control of the potential loss size - ES based multiplier

Gap Risk - control of the probability of a potential loss - VaR based multiplier

▸ Multiplier

- Given a confidence level $1-\alpha$, the insurance condition, i.e., portfolio value is above floor, is guaranteed, Föllmer and Leukert (1999),

$$\mathrm{P}\left(C_t \geq 0,\ \forall t \leq T\right) \ \geq \ 1-\alpha$$

- Equivalently, (set time-varying multiplier)

$$\mathrm{P}\left(m_t \leq \left(-r_{t+1}^{-}\right)^{-1},\ \forall t \leq T-1\right) \ \geq \ 1-\alpha$$

Multiplier

▸ Portfolio Protection ▸ Gap risk

- Gap risk: control of the probability of a potential loss ▸ Details
  Multiplier m_t with quantile - Ameur and Prigent (2014)

$$m_{t,q_\alpha} = |VaR_\alpha(r_{t+1})|^{-1}$$

- Gap risk: control of the potential loss size
  Multiple m_t with expected shortfall - Hamidi et al. (2014)

$$m_{t,\tau} = \left|ES_{e_{t,\tau}}\right|^{-1}$$

Multiplier Density

Multiplier Dynamics

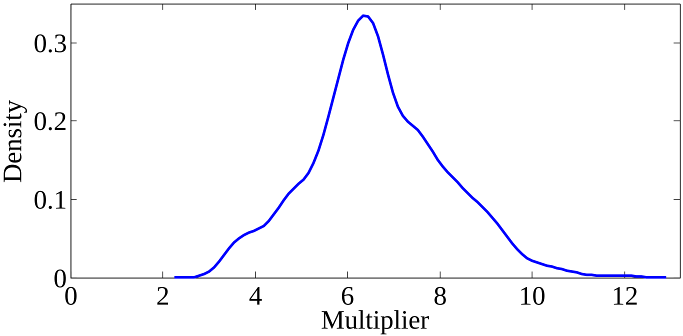


Figure 16: Kernel density estimate of the multiplier $m_{t,\tau}$ for DAX index returns based on ICARE ($r = 1$ and $\tau = 0.05$) from 20060103-20141231

CARE-based one-year rolling

▸ Performance

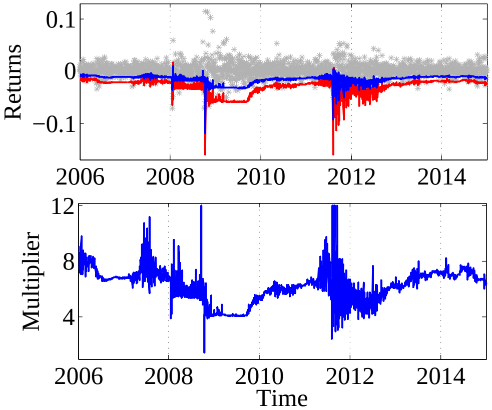
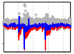


Figure 17: Estimated expectile and expected shortfall by CARE based one-year fixed rolling window (upper panel), and the corresponding multiplier (lower panel) for DAX index returns from 20060103 to 20141231



CAViaR-based one-year rolling

Performance CAViaR

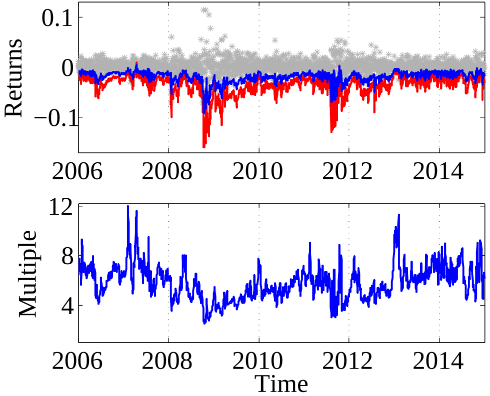
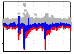


Figure 18: Estimated VaR ($\alpha = 0.065$) and expected shortfall by CAViaR - based one-year rolling (upper panel), and the corresponding multiplier (lower panel) for DAX from 20060103 to 20141231



Portfolio value and target floor ▸ Performance

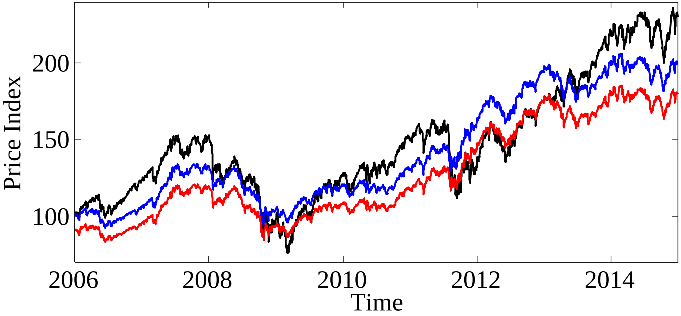
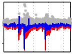


Figure 19: Portfolio value: (a) DAX index (black), (b) $m_{t,\tau}$ - ICARE ($r = 1$ and $\tau = 0.05$), (c) the corresponding target floor F_t^s , from 20060103-20141231.



Portfolio Protection

▸ Motivation ▸ Portfolio Protection

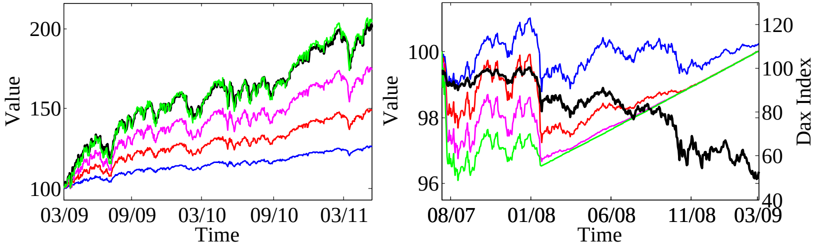


Figure 20: Portfolio value: (a) DAX index, (b) $m = 3$, (c) $m = 6$, (d) $m = 9$, (e) $m = 12$ on DAX index in a bull market from 20090309-20110510 (left panel, 567 observations) and in a bear market from 20070716-20090306 (right panel, 431 observations).

Parameter Dynamics ▸ Motivation

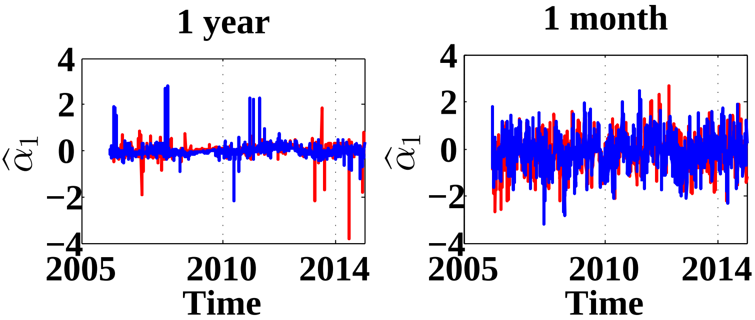


Figure 21: Estimated $\alpha_{1,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations ▸ more parameters

Parameter Dynamics

▸ Motivation

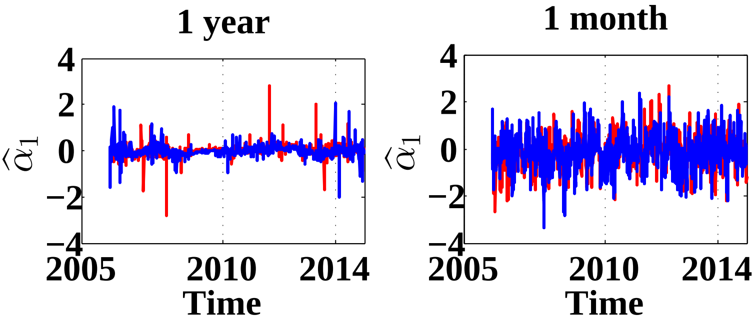
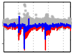


Figure 22: Estimated $\alpha_{1,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations ▸ more parameters



Parameter Distributions

Motivation

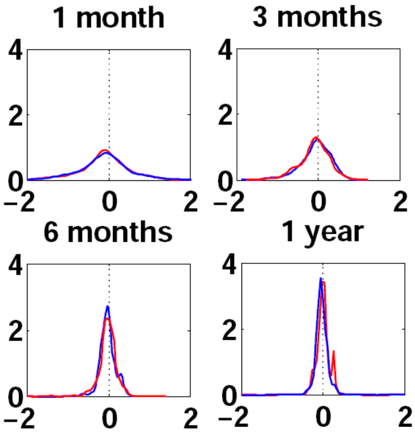


Figure 23: Kernel density estimates of $\alpha_{1,0.05}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

Parameter Distributions

Motivation

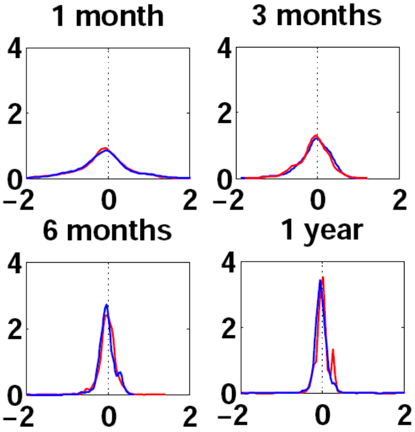


Figure 24: Kernel density estimates of $\alpha_{1,0.01}$ for DAX and FTSE100 using 20, 60, 125 or 250 observations

Parameter Dynamics

▸ Parameter Dynamics

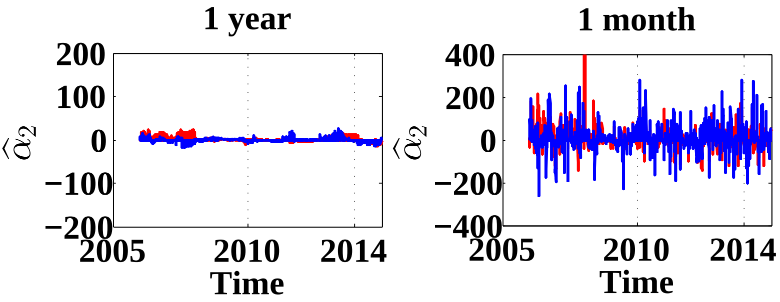


Figure 25: Estimated $\alpha_{2,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

Parameter Dynamics

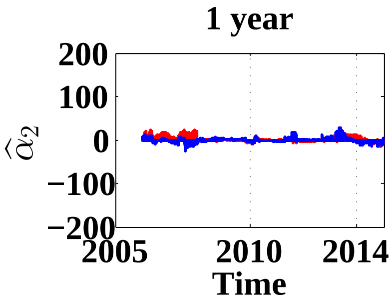


Figure 26: Estimated $\alpha_{2,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

▸ Parameter Dynamics

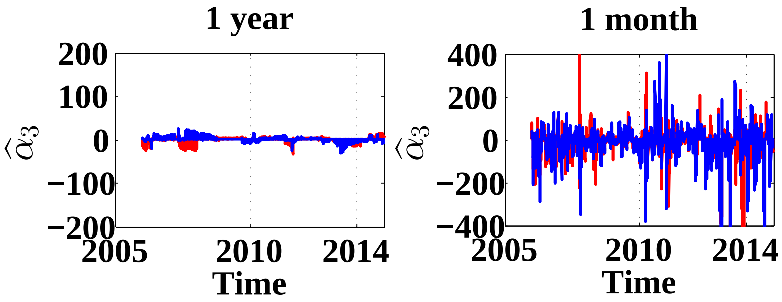


Figure 27: Estimated $\alpha_{3,0.05}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations

Parameter Dynamics

Parameter Dynamics

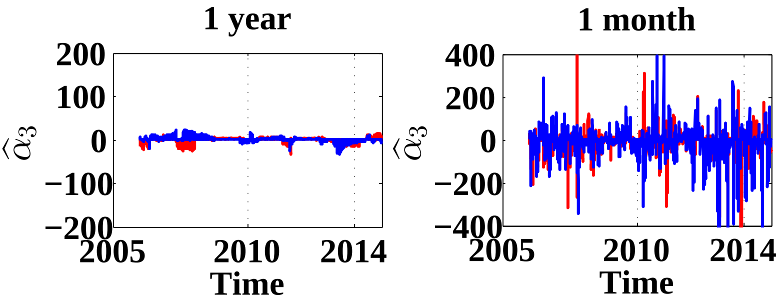


Figure 28: Estimated $\alpha_{3,0.01}$ for DAX and FTSE100 using 20 (1 month) or 250 (1 year) observations